Calculus

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[1] Based on the book of George F. Simons, *Calculus with Analytic Geometry*

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LIMITS

INTUITION

Limit Intuition: Sometimes it is not possible to work out what the value of a function is, it might be indeterminate. So instead we work out the value as we get closer and closer but without actually being 'there'.

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Example:

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We can get $f(x)$ as close to L 'as we want' by getting x sufficiently close to a.

LIMITS

- **Approach from the left/right:** functions need checking the limit from both sides to make sure it actually exists
  - Approach from the left: $\lim_{x\to a^-} f(x)$
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- **Approach from the left/right:** functions need checking the limit from both sides to make sure it actually exists
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  - Approach from the right: $\lim_{x \to a^+} f(x)$

- **Existence:** A limit L exists if the limit from the left is the same that the one from the right.

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x) \text{ for } a \neq \pm\infty$$

If the function is defined only over an interval, the extrema points are only needed to check one of the sides.

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Example: Consider $\lim_{x \to 0} \frac{x}{|x|}$

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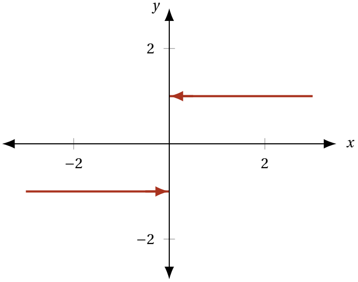

$\lim_{x\to a} \frac{x}{|x|}$

Properties of limits: or limits of combined functions. Now define:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M$$

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Then the properties are:

$$\begin{array}{rcccl}
\lim_{x\to c} f(x)+g(x) & = & \lim_{x\to c} f(x)+\lim_{x\to c} g(x) & = & L+M \\
\lim_{x\to c} f(x)-g(x) & = & \lim_{x\to c} f(x)-\lim_{x\to c} g(x) & = & L-M \\
\lim_{x\to c} f(x)\cdot g(x) & = & \lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x) & = & L\cdot M \\
\lim_{x\to c} f(x)/g(x) & = & \lim_{x\to c} f(x)/\lim_{x\to c} g(x) & = & L/M \\
\lim_{x\to c} kf(x) & = & k\lim_{x\to c} f(x) & = & k\cdot L
\end{array}$$

LIMITS

Exercise: consider the following two limits

$$\lim_{x\to 3} 7x - 6 = L \text{ and } \lim_{x\to 0} \frac{5}{x-1} = M$$

Work out $L + M$, $L - M$,$L \cdot M$ and L/M

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Solution: First work out each limit individually:

$$L = \lim_{x\to 3} 7x - 6 = 15 \text{ and } M = \lim_{x\to 0} \frac{5}{x-1} = -5$$

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Then perform the operations by just substituting the values

1. $M+L = 15-5 = 10$
2. $M-L = 15+5 = 20$
3. $M\cdot L = 15\cdot(-5) = -75$
4. $M/L = 15/(-5) = -3$

LIMITS

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But don't be fooled by the "=". We cannot actually get to infinity, but in "limit" language the limit is infinity (which is really saying the function is limitless).

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Limits at infinity (Horizontal asymptotes): it is the limit of a function as x approaches infinity.

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Examples:

- Rational
- Radical
- Trigonometric
- Difference

LIMITS

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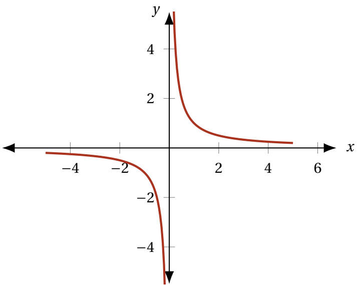

$\lim_{x\to 0^\pm} \frac{1}{x}$ and $\lim_{x\to\infty} \frac{1}{x} = 0$

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Continuity Definition: a function $f(x)$ is said to be continuous if and only if

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

and

$$f(x) = L$$

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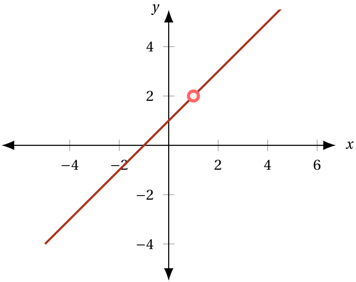

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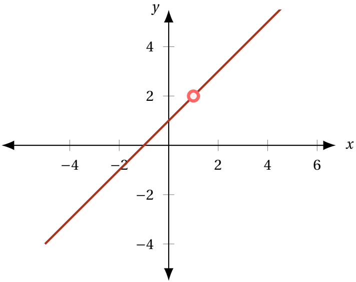

And the function is said to have a removable discontinuity.

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The concept of the **derivative** is of special importance in economics. It allow us to work out the **rate of change** of one variable with respect to other.

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Concepts like **productivity**, **marginal cost** or **marginal utility** are direct applications of the concept of derivative.

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Concepts like **productivity**, **marginal cost** or **marginal utility** are direct applications of the concept of derivative.

Also, they will become quite handy when doing **comparative statics**.

DERIVATIVES

INTUITION

But, what a derivative really is?

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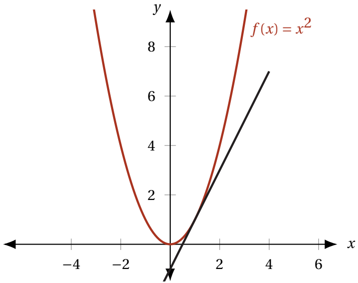

Derivative of x^2 at $(1,1)$

DERIVATIVES

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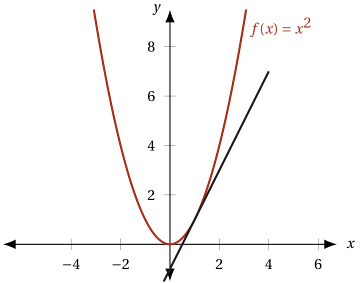

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- The slope of a function

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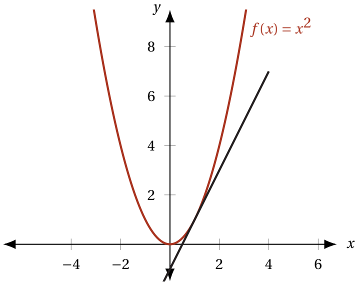

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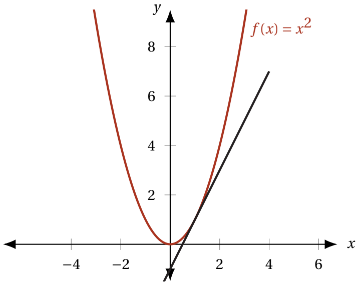

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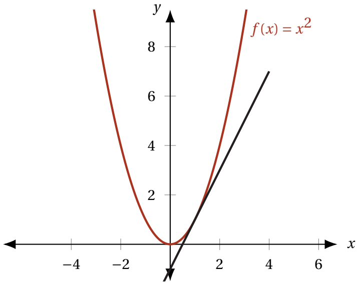

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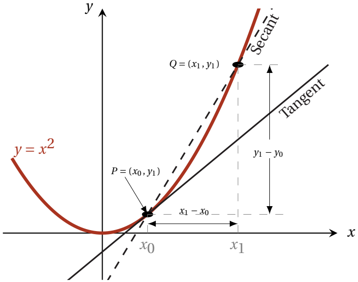
At a point

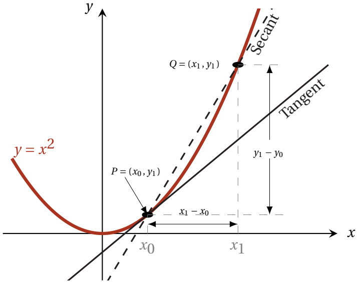
How to calculate the slope of the tangent at $P = (x_0, y_0)$



DERIVATIVES
INTUITION

How to calculate the slope of the tangent at $P = (x_0, y_0)$

1. Choose a point $P = (x_0, y_0)$



DERIVATIVES

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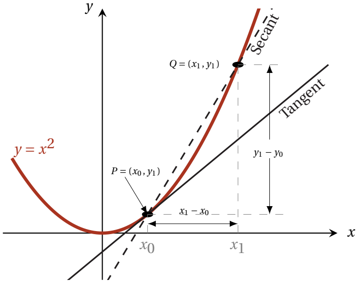

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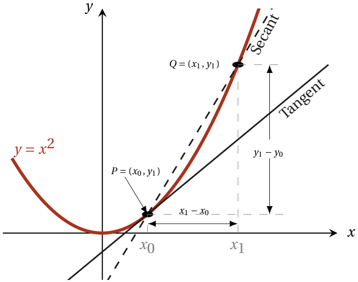

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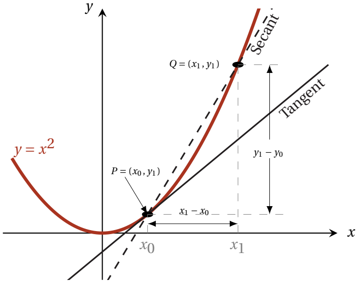
4. Take the limit as $Q \to P$



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$$m_{sec} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0}$$

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- Take the limit

$$m = \lim_{P \to Q} m_{sec} = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0}$$

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- Take the limit

$$m = \lim_{P \to Q} m_{sec} = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0}$$

WARNING!!: at $x_1 = x_0$ the slope is not defined: $m_{sec} = \frac{0}{0}$, that's why we take the limit.

We must think of x_1 as coming very close to x_0 but *remaining distinct from it*

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Solving the limit:

$$\begin{aligned}\lim_{x_1\to x_0}\frac{y_1-y_0}{x_1-x_0} &= \lim_{x_1\to x_0}\frac{x_1^2-x_0^2}{x_1-x_0} && \text{Remember that } y=x^2\\ &= \lim_{x_1\to x_0}\frac{(x_1+x_0)(x_1-x_0)}{x_1-x_0} && \text{Factor the expression}\\ &= \lim_{x_1\to x_0} x_1+x_0 && \text{Cancel out}\\ &= 2x_0\end{aligned}$$

Since our reference is x_0, we prefer to take small changes like Δx close to the point than other nearby point like x_1.

DERIVATIVES

DELTA NOTATION

Since our reference is x_0, we prefer to take small changes like Δx close to the point than other nearby point like x_1.

$\Delta x = x_1 - x_0$: is the change in x going form the first value to the second or alternatively: $x_1 = x_0 + \Delta x$ adding a small amount to the first value.

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Previous example: Re writing m_{sec}

$$m_{sec} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}$$

$x_1 \to x_0$ is equivalent to $\Delta x \to 0$

solving the numerator:

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$$(x_0 + \Delta x)^2 - x_0 = x_0^2 + 2x_0\Delta x + (\Delta x)^2 - x_0^2 \quad \text{Expanding the binomial}$$

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$$\begin{aligned}(x_0+\Delta x)^2 - x_0 =& x_0^2 + 2x_0\Delta x + (\Delta x)^2 - x_0^2 && \text{Expanding the binomial} \\ =& 2x_0\Delta x + (\Delta x)^2 && \text{Cancelling terms}\end{aligned}$$

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=& \Delta x(2x_0 + \Delta x) && \text{factorising}
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And m_{sec} becomes: $m_{sec} = 2x_0 + \Delta x$, taking the limit:

$$m = \lim_{\Delta x \to 0} 2x_0 + \Delta x = 2x_0$$

Definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

DERIVATIVES

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2. Divide by Δx to form the *difference quotient*: $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

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Procedure to compute derivatives:

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2. Divide by Δx to form the *difference quotient*: $\frac{f(x+\Delta x)-f(x)}{\Delta x}$
3. Evaluate the limit of the difference quotient as $\Delta x \to 0$

Example: $y = x^3$

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STEP 1: Operate the numerator till you factorise Δx

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$$\begin{aligned} f(x+\Delta x) - f(x) &= (x+\Delta x)^3 - x^3 \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 \end{aligned}$$

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STEP 2: Divide by Δx

Example: $y = x^3$

STEP 1: Operate the numerator till you factorise Δx

$$\begin{aligned} f(x+\Delta x) - f(x) &= (x+\Delta x)^3 - x^3 \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 \\ &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ &= \Delta x(3x^2 + 3x\Delta x + (\Delta x)^2) \end{aligned}$$

STEP 2: Divide by Δx

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$$

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STEP 3: Evaluate the limit

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STEP 3: Evaluate the limit

$$f'(x) = \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 = 3x^2$$

DERIVATIVES

NOTATION

All of these symbols are equivalent:

$$y' \qquad \frac{dy}{dx} \qquad f'(x) \qquad \frac{df(x)}{dx} \qquad \frac{d}{dx}f(x) \qquad D_x(f(x))$$

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Why the fractions?

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$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

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$$\left.\frac{dy}{dx}\right|_{x=x_0}$$

DERIVATIVES

NOTATION

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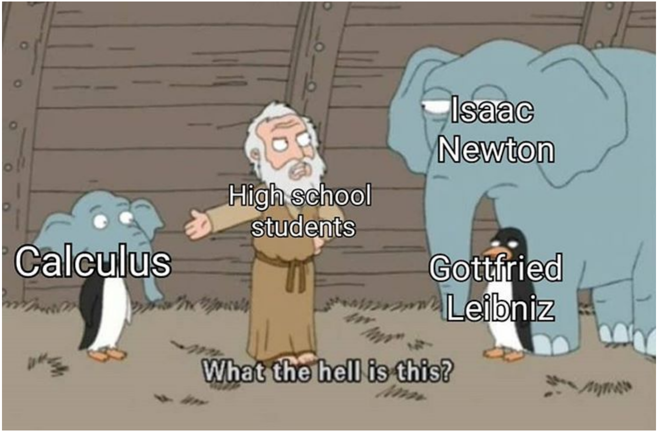

DERIVATIVES

COMPUTATION

CONSTANT: $y = c$

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Proof:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

DERIVATIVES

COMPUTATION

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$$\frac{dy}{dx} = \lim_{(\Delta x) \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$ Substitute

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$$= \lim_{\Delta x \to 0} \frac{\left(x^n + n x^{n-1}\Delta x + \cdots + n x (\Delta x)^{n-1} + (\Delta x)^n\right) - x^n}{\Delta x} \qquad \text{Expand } (x+\Delta x)^n$$

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$$= \lim_{\Delta x \to 0} \left(n x^{n-1} + \frac{n(n-1)}{2!} x^{n-2}\Delta x + \cdots + n x(\Delta x)^{n-2} + (\Delta x)^{n-1} \right) \qquad \text{Evaluate}$$

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Proof:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{(\Delta x)\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} && \text{Substitute} \\
&= \lim_{\Delta x\to 0} \frac{\left(x^n + nx^{n-1}\Delta x + \cdots + nx(\Delta x)^{n-1} + (\Delta x)^n\right) - x^n}{\Delta x} && \text{Expand } (x+\Delta x)^n \\
&= \lim_{\Delta x\to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \cdots + nx(\Delta x)^{n-1} + (\Delta x)^n}{\Delta x} && \text{Cancel terms} \\
&= \lim_{\Delta x\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}\right) && \text{Evaluate} \\
&= nx^{n-1}
\end{aligned}$$

CONSTANT TIMES A FUNCTION: $y = cf(x)$

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$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} && \text{Substitute} \\ &= \lim_{\Delta x \to 0} \frac{c\left(f(x+\Delta x) - f(x)\right)}{\Delta x} && \text{Factor } c \\ &= c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} && \text{Evaluate} \end{aligned}$$

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SUM OF FUNCTIONS: $y = f(x) + g(x)$

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SUM OF FUNCTIONS: $y = f(x) + g(x)$

$$\frac{d}{dx}\big(f(x)+g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\big(f(x+\Delta x)+g(x+\Delta x)\big)-\big(f(x)+g(x)\big)}{\Delta x} && \text{Substitute}\\
&= \lim_{\Delta x\to 0}\frac{\big(f(x+\Delta x)-f(x)\big)+\big(g(x+\Delta x)-g(x)\big)}{\Delta x} && \text{Factorise}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x} && \text{Limit rules}\\
&= f'(x)+g'(x)
\end{aligned}$$

PRODUCT RULE: $y = f(x) \cdot g(x)$

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$\frac{d}{dx}\left[f(x) \cdot g(x)\right]$

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$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x)}{\Delta x}$$

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Add and subtract $f(x+\Delta x)g(x)$

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Re arrange

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Limit rules

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$$= \underbrace{\lim_{\Delta x \to 0} f(x+\Delta x)}_{f(x)} \cdot \underbrace{\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}}_{g'(x)} + \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}_{f'(x)} \cdot \underbrace{\lim_{\Delta x \to 0} g(x)}_{g(x)} =$$

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\frac{d}{dx}\left[f(x)\cdot g(x)\right] \\
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&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} && \text{Add and subtract } f(x+\Delta x)g(x) \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\left[g(x+\Delta x) - g(x)\right] + \left[f(x+\Delta x) - f(x)\right]g(x)}{\Delta x} && \text{Re arrange} \\
&= \lim_{\Delta x \to 0} f(x+\Delta x)\cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\cdot g(x) && \text{Limit rules} \\
&= \underbrace{\lim_{\Delta x \to 0} f(x+\Delta x)}_{f(x)} \cdot \underbrace{\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}}_{g'(x)} + \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}_{f'(x)} \cdot \underbrace{\lim_{\Delta x \to 0} g(x)}_{g(x)} = \\
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DERIVATIVES

COMPUTATION

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Notice that for a continuous function $g(x)$ at a point:

$$as\ \Delta x \to 0 \Rightarrow \Delta g(x) \to 0$$

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Then the result follows:

$$\frac{df(g(x))}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x} = \lim_{\Delta g \to 0} \frac{\Delta f}{\Delta g} \cdot \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

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Proof:

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Apply the product rule, for the second term use the power rule for $g(x)^{-1}$ then apply the chain rule.

DERIVATIVES

IMPLICIT DIFFERENTIATION

Up to now all the functions have been of the form $y = f(x)$

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Example: take x to be a function of y, such that $x = g(y)$ and $x' = \frac{dx}{dy}$.

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And taking into account that y is a function of x all the way along, the next two expressions are equivalent

$$y = x^{\frac{p}{q}} \Leftrightarrow y^q = x^p$$

IMPLICIT DIFFERENTIATION

Assuming y depends on x and using implicit differentiation on both sides of $y^q = x^p$:

DERIVATIVES

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DERIVATIVES

COMPUTATION

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Now let's assume that $\exists! a = e | M(e) = 1$, Then:

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The proof for the $log_a x$ in any base a is identical to the $\ln x$

INCREASE: What means for a function to be increasing?

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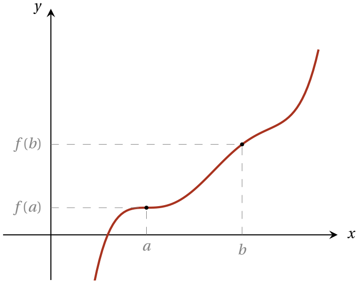

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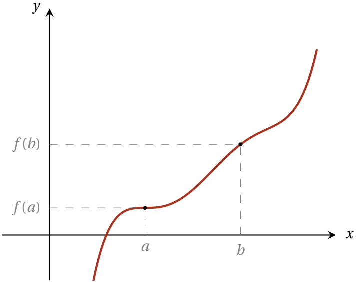

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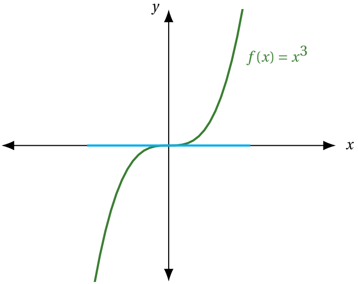

The derivative $f'(0) = 0$ but the function is increasing at that point.

MAXIMUM/MINIMUM: Where does the function attains its local maxima and minima?

DERIVATIVES

APPLICATIONS

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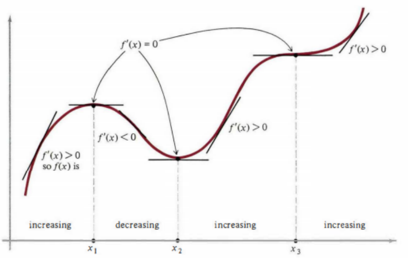
$$\text{if } f'(x_0) = 0 \Rightarrow f(x_0) \text{ is a critical point}$$



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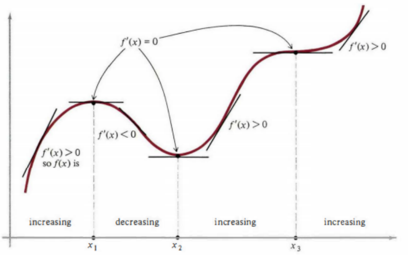

WHACHT OUT!!! $f'(x_0) = 0$ does not mean that we are in a maximum or a minimum at x_0. I could be an inflection point

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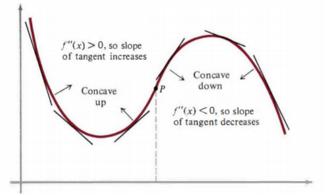

CONCAVITY AND POINTS OF INFLECTION: In what direction does the curve of the function bends?

- If $f''(x_0) > 0 \Rightarrow$
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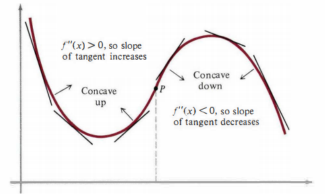

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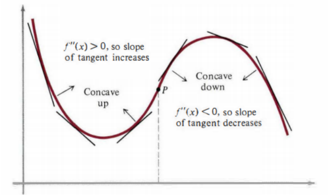

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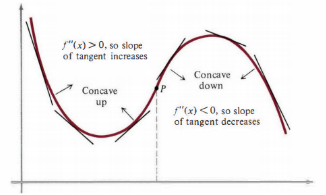

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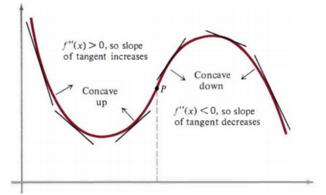

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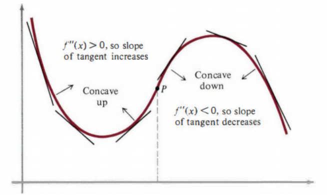

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- If $f''(x_0) = 0 \Rightarrow f(x_0)$ could be max, min or an inflection point

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$$f(x+dx) \approx f(x) + f'(x)\{(x+dx)-x\} \text{, for } x \approx x+dx$$

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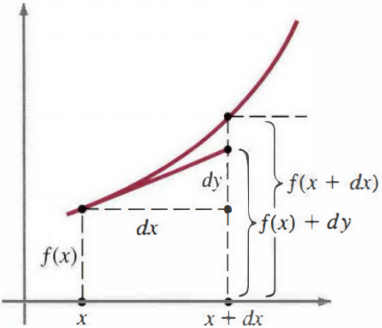

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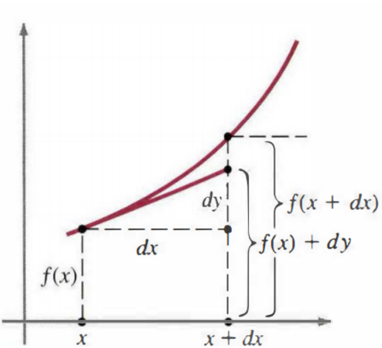

Play with this example to see how good an approximation can get as we get very near to the point.

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Theorem: If $f(x)$ and $g(x)$ are both equal to zero at $x = a$ and have derivatives there, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

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$$\lim_{x\to 2}\frac{3x^2-7x+2}{x^2+5x-14}=\frac{0}{0}$$

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Or using L'Hospital's rule:

$$\lim_{x\to 2}\frac{3x^2-7x+2}{x^2+5x-14}=\frac{0}{0}$$ L'H's rule

$$\lim_{x\to 2}\frac{6x-7}{2x+5}=\frac{5}{9}$$

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INTEGRALS

INTUITION

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Another important concept is that of finding the **area** under the curve, which is called the process of **integration**.

As you have probably seen in your degree, integrals have plenty of applications like calculate the consumer surplus, work out the lifetime utility or derive the income distribution.

For our purposes it will be handy to introduce a **’new’** notation of differentials.
We defined the slope as

$$m = \frac{\Delta y}{\Delta x} so \qquad \Delta y = m \Delta x$$

If we work on increments on the **straight line**, we take the **differential counterpart**, then

$$\Delta y = dy, \Delta x = dx and tuhs dy = m\,dx$$

INTEGRALS

INTUITION

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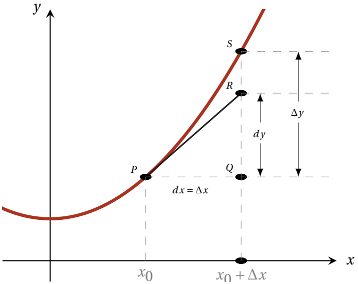
This will allow us to work with differential as if they were **quotients**.



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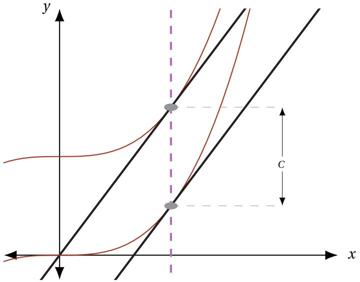
Example:

$$f(x) = 3x^2 \iff F(x) = x^3 + C$$



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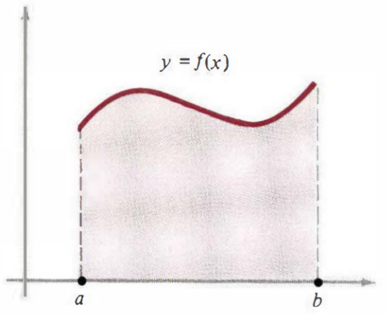

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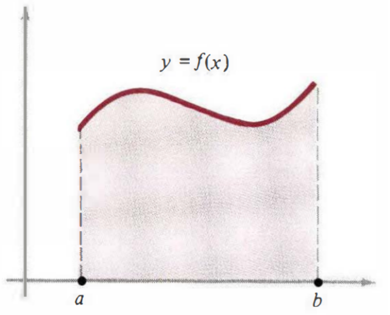

WHACHT OUT!!! Indefinite and definite integrals are two completely different objects, they must not be confused.

INTEGRALS

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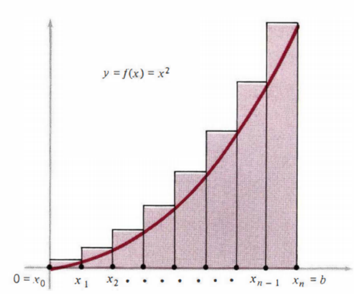

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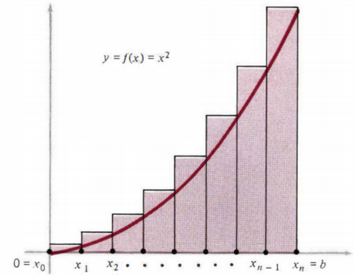

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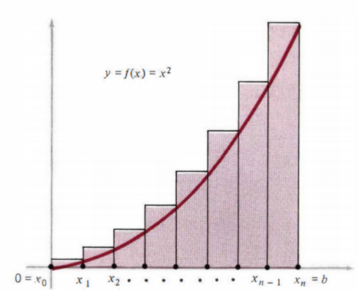

Of course, there is going to be some error, that can be avoided doing the intervals *"as small as possible"*

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\frac{\Delta x}{n}$$

FUNDAMENTAL THEOREM OF CALCULUS II: Let $f(x)$ be a continuous non-negative function in a close interval $[a,b]$. Then:

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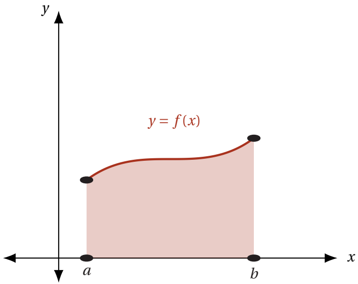

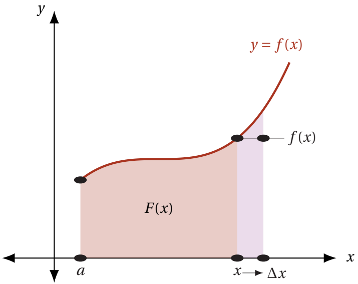

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Taking the limit as $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta F(x)}{\Delta x} = f(x) \Longleftrightarrow F'(x) = f(x)$$

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$$G(x) = \int_a^x f(t)\,dt \overset{byFTCII}{\Longrightarrow} G'(x) = f(x)$$

$$\text{since } G'(x) = f(x) = F'(x), \text{ we have } (G(x) - F(x))' = 0$$

PROOF:

$$\text{then } G(x) - F(x) = C$$

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Then evaluate the function $G(x)$ at $x = b$ and use the value of C above:

$$G(b) = F(b) - F(a) \iff \int_a^b f(t)\,dt = F(b) - F(a)$$

INDEFINITE AND DEFINITE INTEGRALS:

$$\int c\, f(x)\, dx = c \int f(x)\, dx$$

$$\int \left[f(x) + g(x) \right] dx = \int f(x) dx + \int g(x)\, dx$$

DEFINITE INTEGRALS:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

DEFINITE INTEGRALS:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \text{ and } \frac{d}{dx}\int_x^b f(t)dt = -f(x)$$

$$\text{if } f(x) \geq g(x), \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

$$\text{if } f(x) \leq 0, \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \leq 0$$

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

ANTIDERIVATIVE: Some integrals are easy to work out because they are just the opposite operation of the derivative.

$$\int_a^b e^x\,dx = e^x\Big]_a^b + c \qquad \int_a^b \frac{1}{x}\,dx = ln x\Big]_a^b + c$$

$$\int_a^b \sin x\,dx = -\cos x\Big]_a^b + c \qquad \int_a^b \cos x\,dx = \sin x\Big]_a^b + c$$

$$\int_a^b x^n\,dx = \frac{x^{n+1}}{n+1}\Bigg]_a^b + c$$

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Integrating:

$$y = \int_a^b y'dx = \int_a^b f\left(g(x)\right)g'(x)dx$$

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Substituting:

$$F(x) = \int_1^2 \ln x \frac{1}{x}\,dx = \int_{u(1)}^{u(2)} u\,du = \left.\frac{u^2}{2}\right]_0^{\ln 2} = \left.\frac{1}{2}(\ln x)^2\right]_1^2 + C$$

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Integrating:

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INTUITION: the main objective is to make $f(x)$ into something simpler, whilst letting $g(x)$ to remain in something similar or not more complicated.

Example: find the integral of the function $f(x) = x^2 e^x$ in the interval $[0, 1]$

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Integrating by parts:

$$\int_0^1 x^2 e^x \, dx = x^2 e^x \big]_0^1 - \int_0^1 2xe^x \, dx = x^2 e^x \big]_0^1 - 2xe^x \big]_0^1 + 2\int_0^1 e^x \, dx$$
$$= \left(x^2 - 2x + 2\right) e^x \big]_0^1 = e - 2$$

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INTEGRALS

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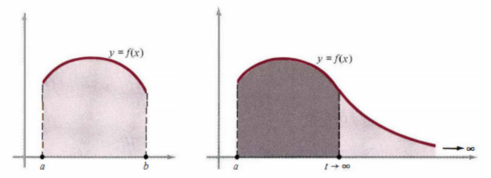
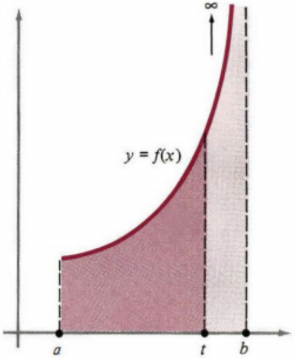


IMPROPER INTEGRALS: are integrals of the form:

$$\int_a^b f(x)\,dx = \lim_{t \to b} \int_a^t f(x)\,dx$$

In which $f(x)$ becomes infinite as x approaches b



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Examples: convergent integrals

$$\int_0^{\infty} e^{-x}\,dx = -\left[e^{-x}\right]_0^{\infty} = -\lim_{b\to\infty}\left[e^{-x}\right]_0^{b} = -0+1 = 1+C$$

$$\int_0^{1} x^{-\frac{1}{2}}\,dx = 2\left[x^{\frac{1}{2}}\right]_0^{1} = 2\,[1-0] = 2+C$$

Examples: divergent integrals

$$\int_0^{\infty} \frac{1}{x} dx = \ln x]_1^{\infty} = \ln \infty - \ln 1 = \infty - 0 = \infty$$

$$\int_0^1 x^{-2} dx = -\left[\frac{1}{x}\right]_0^1 = -1 + \lim_{x \to 0^+} \frac{1}{x} = -1 + \infty = \infty$$

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POWER SERIES

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Example:

$$\sum x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \text{ for} x < |1|$$

POWER SERIES

As well as polynomials, that are finite, power series share some interesting characteristics. It can be said that within the radius of convergence:

- Power series are continuous
- Are differentiable
- Are integrable

TAYLOR POWER SERIES: we have seen that power series are functions in their own right, some of them with a close form solution, such as: $\sum x^n = \frac{1}{1-x}$.

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Look at the gif of $\ln(1+x)$ for some intuition

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Assume we have any $f(x)$ and we would like to write in the form of a power series, i.e.

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From this expression we can take infinitely many derivatives:

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From this expression we can take infinitely many derivatives:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

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Assume we have any $f(x)$ and we would like to write in the form of a power series, i.e.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

From this expression we can take infinitely many derivatives:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$
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$$f^n(x) = n!a_n + \text{Terms containing } x \qquad f^n(0) = n!a_n \quad \Rightarrow a_n = \frac{1}{n!} f^n(0)$$

Substituting back into the original equation:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^3(0)}{3!}x^3 + ... + \frac{f^n(0)}{n!}x^n + ...$$
$$= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$

Example: take $\ln(1 + x)$

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Example: $\ln(1+x)$
Substituting back into Taylor's formula:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{n+1}$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Look at the gif for $\ln(1+x)$

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MULTIVARIATE CALCULUS

INTRODUCTION

Many functions do not depend only on one variable but in an undefined number of them, e.g.:

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Is a function that depends only on x and y. Of course a function might have any number of variables:

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This specific arrange of variables is called a **vector**. As such, we can define bold **x** as this vector, hence:

$$\mathbf{x} = (x_1, x_2, ..., x_n)$$

DOMAIN: the domain is all the points $P = (x_{1_0}, x_{2_0}, \ldots, x_{n_0})$ in the n-dimensional space for which the function $z = f(\mathbf{x})$ is defined

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This function is not define for all values where $x^2 + y^2 \geq 9$

LEVEL CURVES

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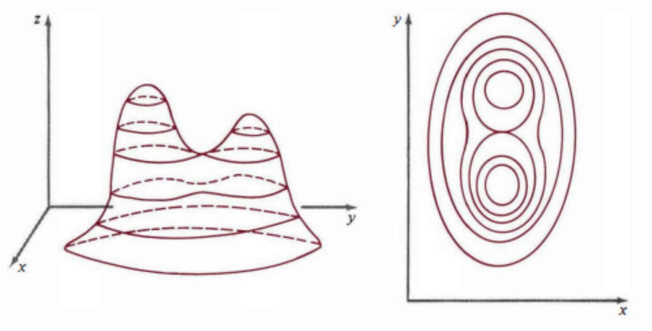
The collection of level curves is called the **contour-map**



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And in general:

$$\frac{\partial z}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_i + \Delta x_i, \mathbf{x}_{-i}) - f(\mathbf{x})}{\Delta x_i} = f_{x_i}(\mathbf{x})$$

Where $\mathbf{x}_{-i}$ are all other variables different from x_i

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$$

NOTATION: $\frac{\partial z}{\partial x}$ this limit (if it exist) is the *partial derivative of* z *w.r.t.* x. The most common notations are:

$$
\frac{\partial z}{\partial x}, \quad z_x, \quad \frac{\partial f}{\partial x}, \quad f_x, \quad f_x(x,y)
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MULTIVARIATE CALCULUS

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More interestingly, usually $f_{xy} = f_{yx}$

Example:

$$f_x = 4x^3 + 6xy^3 - \frac{2}{x} \qquad f_{yx} = 18xy^2$$

$$f_y = 9x^2y^2 - \frac{1}{y} \qquad f_{xy} = 18xy^2$$

TANGENT PLANE: The concept of tangent plane to a surface corresponds to the concept of tangent line to a curve. So the tangent plane of a surface at a point is the plane that "best approximates" the surface at that point.

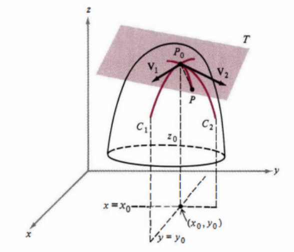

Figure: Tangent plane

Tangent line	Tangent plane
$m(x-x_0)+(y-y_0)=0$	$a(x-x_0)+b(y-y_0)+(z-z_0)=0$
$f'(x_0)(x-x_0)+\big(f(x)-f(x_0)\big)=0$	$f_x(x-x_0)+f_y(y-y_0)+\big(f(x,y)-f(x_0,y_0)\big)=0$

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CHAIN RULE

Let $w = f(x, y)$ be a differentiable function in a closed interval. Let also $x = g(t)$ and $y = h(t)$ be continuous functions in the same interval. Then

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And in general for $w = f(\mathbf{x})$:

$$\frac{\partial f(\mathbf{x})}{\partial t} = \frac{\partial f(\mathbf{x})}{\partial x_1}\frac{\partial x_1}{\partial t} + ... + \frac{\partial f(\mathbf{x})}{\partial x_n}\frac{\partial x_n}{\partial t}$$

IMPLICIT FUNCTION THEOREM

THEOREM

THEOREM: Let $F(x,y)$ have continuous partial derivatives throughout some neighbourhood of a point (x_0, y_0), assume also that $F(x_0, y_0) = c$ and $F_y(x_0, y_0) \neq 0$. Then there is an interval I about x_0 with the property that there exists exactly one differentiable function $y = f(x)$ defined on I such that $y_0 = f(x_0)$ and:

$$F\left[x, f(x)\right] = c$$

Further, the derivative of this function is given by the formula

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

and is therefore continuous.

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Differentiate $F\left[x, f(x)\right] = c$ w.r.t x using the chain rule

$$\frac{\partial F\left[x, f(x)\right]}{\partial x} = F_x + F_y \frac{\partial y}{\partial x} = 0$$

IMPLICIT FUNCTION THEOREM

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Proof: for the second statement

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$$\frac{\partial F\left[x, f(x)\right]}{\partial x} = F_x + F_y \frac{\partial y}{\partial x} = 0$$

Solving for $\frac{\partial y}{\partial x}$ the result follows

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

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Then

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2xy^5 - 2y}{5x^2 y^4 - 2x}$$

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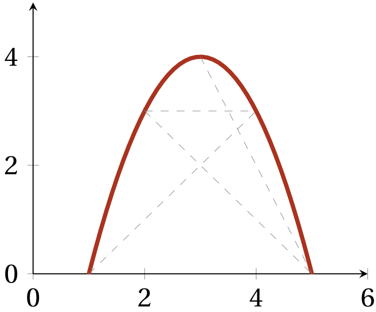
CONVEX AND CONCAVE FUNCTIONS

INTUITION

CONCAVE FUNCTION: is a function where no line segment joining two points on the graph lies **above** the graph at any point.



CONVEX AND CONCAVE FUNCTIONS

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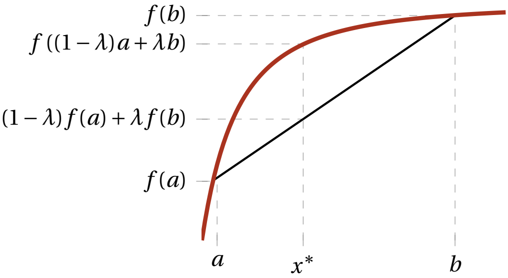
$$f((1-\lambda)a + \lambda b) \geq (1-\lambda) f(a) + \lambda f(b)$$



where $x* = (1-\lambda)a + \lambda b$

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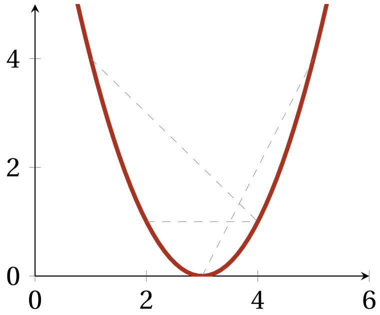

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$$f((1-\lambda)a + \lambda b) \leq (1-\lambda) f(a) + \lambda f(b)$$

CONVEX AND CONCAVE FUNCTIONS

JENSEN'S INEQUALITY

A function $f(x)$ of a single variable defined on the interval I is **concave** if and only if $\forall n \geq 2$:

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

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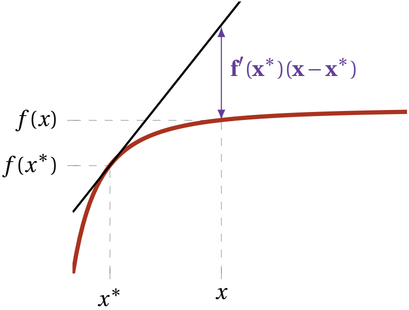
CONVEX AND CONCAVE FUNCTIONS

DIFFERENTIABLE FUNCTIONS

DEFINITION: The differentiable function $f(x)$ of a single variable defined on an open interval I is **concave** on I if and only if:

$$f(x) - f(x^*) \leq f'(x^*)(x - x^*)$$



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Play with this graph

CONVEX AND CONCAVE FUNCTIONS

TWICE-DIFFERENTIABLE FUNCTIONS

PROPOSITION: A twice-differentiable function $f(x)$ of a single variable defined on the interval I is:

- **Concave:** if and only if $f''(x) \leq 0$ for all x in the interior of I
- **Convex:** if and only if $f''(x) \geq 0$ for all x in the interior of I

INTUITION: For a concave (convex) function, the slope of the tangent line to a point becomes lesser as we move along the x-axis